CONSTRUCTIONS FOR CYCLIC SIEVING PHENOMENA

ANDREW BERGET, SEN-PENG EU, AND VICTOR REINER

ABSTRACT. We show how to derive new instances of the cyclic sieving phenomenon from old ones via elementary representation theory. Examples are given involving objects such as words, parking functions, finite fields, and graphs.

1. INTRODUCTION

The *cyclic sieving phenomenon (CSP)* was introduced in [12], generalizing Stembridge's $q = -1$ *phenomenon* [18]. The CSP pertains to a finite set X, carrying the permutation action of a finite abelian group written explicitly as a product $\mathbf{C} := C_1 \times \cdots \times C_m$ of cyclic groups C_i, and a polynomial $X(\mathbf{u}) := X(u_1, \ldots, u_m)$ in $\mathbb{Z}[\mathbf{u}]$, often a generating function for the elements of X according to some natural statistic(s). One says that the triple $(X, X(\mathbf{u}), \mathbf{C})$ *exhibits the CSP* if after choosing embeddings[1] of groups $\omega_i : C_i \hookrightarrow \mathbb{C}^\times$, one has for every $\mathbf{c} = (c_1, \ldots, c_m)$ in $\mathbf{C}$ that the cardinality of its fixed point set $X^{\mathbf{c}} := \{x \in X : \mathbf{c}(x) = x\}$ is given by

$$|X^{\mathbf{c}}| = [X(\mathbf{u})]_{u_i=\omega_i(c_i)}\,.$$

In other words, the generating function $X(\mathbf{u})$ not only has the usual property that its evaluation with all $u_i = 1$ gives the cardinality $|X|$, but furthermore, its evaluation at appropriate roots-of-unity carries all the numerical information about the $\mathbf{C}$-orbit structure on X.

For example, one can equivalently rephrase the CSP (see [2, Proposition 3.1]) as a combinatorial interpretation for the coefficients in the unique expansion

$$X(\mathbf{u}) \equiv \sum_{\substack{\mathbf{d}=(d_1,\ldots,d_m)\\ 0\le d_i<|C_i|}} a_{\mathbf{d}}\ \mathbf{u}^{\mathbf{d}} \mod (u_1^{|C_1|}-1, \ldots, u_m^{|C_m|}-1).$$

Specifically, the CSP asserts that the constant term $a_{(0,0,\ldots,0)}$ counts the total number of $\mathbf{C}$-orbits on X, and more generally, the coefficient $a_{\mathbf{d}}$ counts the number of $\mathbf{C}$-orbits for which the pointwise-$\mathbf{C}$-stabilizer subgroup of any element in the orbit lies in the kernel of the degree one character

$$\omega^{\mathbf{d}} := \prod_{i=1}^{m} \omega_i^{d_i}. \tag{1}$$

Date: January 12, 2011.

1991 *Mathematics Subject Classification.* 05E10, 05E18, 13A50.

Key words and phrases. cyclic sieving phenomena, exterior power, symmetric power, tensor power, Schur-Weyl duality, fake degree, Frobenius, parking function.

A. Berget is partially supported by NSF grant DMS-0636297. S.-P. Eu is partially supported by the National Science Council of Taiwan under grant NSC 98-2115-M-390-002-MY3. V. Reiner was supported by NSF grant DMS-0601010.

[1]See Appendix A below for a discussion of the dependence on this choice of embeddings, and on the choice of decomposition $\mathbf{C} := C_1 \times \cdots \times C_m$.

In [2, 12], various instances of CSP's were shown, sometimes proven via representation theory. The point of the current paper is to show how this viewpoint, combined with the standard multilinear constructions from representation theory of *tensor products* $V_1 \otimes V_2$, *symmetric powers* $\mathrm{Sym}^k(V)$, *exterior powers* $\bigwedge^k(V)$, and *tensor powers* $V^{\otimes \ell} := V \otimes \cdots \otimes V$, allow one to automatically construct new CSP's from old ones. Section 2 develops these constructions, and uses them to derive some interesting new CSP's. We remark that a somewhat different use of representation theoretic constructions to derive new CSP's appears in Westbury [19].

We illustrate our results in the remainder of this introduction, including some explicit examples. For the sake of stating these, recall a notion from [12]: A cyclic group acts *nearly freely* on a finite set if either all orbits have the same size, or if there is a unique singleton orbit and all non-singleton orbits have the same size. We will also need a few notations. For a positive integer n, define

$$[n] := \{1, 2, \ldots, n\}$$
$$[n]_u := 1 + u + u^2 + \cdots + u^{n-1}$$

and for a polynomial $f(x_1, \ldots, x_n)$ in a variable set $\mathbf{x} = (x_1, \ldots, x_n)$, its *principal u-specialization* is $f(1, u, u^2, \ldots, u^{n-1})$.

1.1. **Words.** Consider the set $[n]^\ell$ of words $w_1 w_2 \cdots w_\ell$ of length ℓ with letters in the alphabet $[n]$. Given such a word w, its *inversion number* $\mathrm{inv}(w)$ is the number of pairs (i, j) with $1 \leq i < j \leq \ell$ for which $w(i) > w(j)$, while its *major index* $\mathrm{maj}(w)$ is the sum of all positions i in the range $1 \leq i < n$ for which $w(i) > w(i+1)$. A famous result of MacMahon (see [4]) asserts that these two statistics are equidistributed as one runs over all rearrangements of a fixed word, so that one has an equality

$$f(\mathbf{x}, t) := \sum_{w \in [n]^\ell} x_{w_1} \cdots x_{w_\ell} \, t^{\mathrm{maj}(w)} = \sum_{w \in [n]^\ell} x_{w_1} \cdots x_{w_\ell} \, t^{\mathrm{inv}(w)}. \tag{2}$$

Theorem 1. *Let $X = [n]^\ell$ be permuted by $C_1 \times C_2$ in which C_1 is a cyclic group acting nearly freely on the letter values $[n]$, and C_2 is a cyclic group acting nearly freely on the word positions $[\ell]$.*

Let $X(u, t)$ be the principal u-specialization in the $\mathbf{x}$-variables of $f(\mathbf{x}, t)$.

Then $(X, X(u, t), C_1 \times C_2)$ exhibits the CSP.

Example. Take $n = 3$, $\ell = 2$, so that $X = [3]^2$. Let $C_1 = \langle c_1 \rangle$ be a cyclic group of order 3 cyclically permuting the letter values [3], and let $C_2 = \langle c_2 \rangle$ be a cyclic group of order 2 swapping the two positions [2] in the words. Then the set $X = [3]^2$ decomposes into these $C_1 \times C_2$-orbits

$$\left\{ \begin{matrix} 12 & 23 & 31 \\ u & u^3 & u^2 t \\ 21 & 32 & 13 \\ ut & u^3 t & u^2 \end{matrix} \right\}, \quad \left\{ \begin{matrix} 11 & 22 & 33 \\ 1 & u^2 & u^4 \end{matrix} \right\}$$

in which each element of X is shown with the term it contributes to $X(u, t)$ just below it. The orbits are arranged tabularly so that C_1, C_2 act cyclically on the row, column indices respectively. The first orbit is $C_1 \times C_2$-free, while in the second orbit c_2 acts trivially.

From the data above (or see Section 4 below) one can calculate

$$X(u, t) = 1 + u + 2u^2 + u^3 + u^4 + t(u + u^2 + u^3)$$

and hence

$$X(u,t) \equiv 2 + 2u + 2u^2 + t(1 + u + u^2) \mod (u^3 - 1, t^2 - 1).$$

Note that in this last expression, the constant term 2 matches the total number of orbits. As an example of the root-of-unity evaluations predicted by the CSP, note that $X(e^{\frac{2\pi i}{3}}, 1) = X(e^{\frac{2\pi i}{3}}, -1) = 0$, corresponding to the fact that neither $(c_1, 1)$ nor (c_1, c_2) fix any elements of X. On the other hand, $X(1, -1) = 3$ counts the elements in the second orbit, which are fixed by $(1, c_2)$.

1.2. **Finite fields.** Theorem 1 combined with the Normal Basis Theorem from Galois theory will turn out to have the following consequence for the action of the Frobenius endomorphism on a finite field $\mathbb{F}_{q^\ell}$ for any prime power q.

Theorem 2. *Let $X = \mathbb{F}_{q^\ell}$ be permuted by $C_1 \times C_2$ in which the cyclic group $C_1 = \mathbb{F}_q^\times$ of order $q-1$ acts via multiplication, and the cyclic group $C_2 = \mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)$ of order ℓ generated by the Frobenius endomorphism acts as usual.*

Let $X(u,t)$ be the same as in Theorem 1, taken with $n := q$.

Then $(X, X(u,t), C_1 \times C_2)$ exhibits the CSP.

Example. Take $q = 3$, $\ell = 2$, so that

$$X = \mathbb{F}_{3^2} = \mathbb{F}_9 = \{0, 1, \pi, \pi^2, \pi^3, \pi^4, \pi^5, \pi^6, \pi^7\}$$

where π is a cyclic generator for the multiplicative group $\mathbb{F}_9^\times \cong \mathbb{Z}_8$. The subfield $\mathbb{F}_3$ embeds in $\mathbb{F}_9$ as $\mathbb{F}_3 = \{0, 1, \beta\}$ where $\beta = \pi^4$, and $C_1 = \mathbb{F}_3^\times = \langle \beta \rangle = \{1, \beta\}$ is a cyclic group of order 2 acting on X by multiplication. The Frobenius map $F : \alpha \mapsto \alpha^3$ generates the Galois group $C_2 = \mathrm{Gal}(\mathbb{F}_{3^2}/\mathbb{F}_3) = \langle F \rangle$ of order two, also acting on X. Then the set X decomposes into these three $C_1 \times C_2$-orbits

$$\left\{ \begin{array}{ccc} \pi & \xleftrightarrow{F} & \pi^3 \\ \beta \updownarrow & & \updownarrow \beta, \\ \pi^5 & \xleftrightarrow{F} & \pi^7 \end{array} \qquad \begin{array}{c} \pi^2 \\ \beta \updownarrow \quad F, \\ \pi^6 \end{array} \qquad {}^{F}\circlearrowleft 1 \xleftrightarrow{\beta} \beta \circlearrowright^{F}, \qquad {}^{F}\circlearrowleft 0 \circlearrowright^{\beta} \right\}$$

The first orbit is $C_1 \times C_2$-free. The second orbit has both β and F acting by swapping the two elements. The third orbit is fixed by F and has its two elements swapped by β. The last orbit is a singleton fixed by both β and by F.

In the example following Theorem 1 we computed

$$X(u,t) = 1 + u + 2u^2 + u^3 + u^4 + t(u + u^2 + u^3)$$

and hence

$$X(u,t) \equiv 4 + 2u + t + 2ut \mod (u^2 - 1, t^2 - 1).$$

Note that in this last expression, the constant term 4 matches the total number of orbits. As an example of the root-of-unity evaluations predicted by the CSP, note that $X(1, -1) = 3$ counting the elements in the third and fourth orbits, which are fixed by $(1, F)$, that $X(-1, 1) = 1$ counting the element in the fourth orbit, which is fixed by $(\beta, 1)$, and that $X(-1, -1) = 3$ counting the elements in the second and fourth orbits, which are fixed by (β, F).

1.3. **Parking functions.** A word of length ℓ in the alphabet $\{1, 2, \ldots\}$ is called a *parking function* (see, *e.g.*, Haiman [5], Kung, Sun and Yan [6], Pak and Postnikov [10]) if the weakly increasing rearrangement $a_1 \leq a_2 \leq \cdots \leq a_\ell$ of its letters has $a_i \leq i$ for all i. Theorem 1 turns out to be closely related to the following result, discussed in Section 3 below.

Theorem 3. *Let X be the set of parking functions of length ℓ, permuted by a cyclic group C acting nearly freely on the set $[\ell]$ of positions.*

Let

$$X(t) := \sum_{w \in X} t^{\mathrm{maj}(w)} = \sum_{w \in X} t^{\mathrm{inv}(w)}.$$

Then $(X, X(t), C)$ exhibits the CSP.

Example. For $\ell = 3$, the set X of parking functions is

$$X = \{111, \quad 112, 121, 211, \quad 113, 131, 311, \quad 122, 212, 221, \\ 123, 132, 213, 231, 312, 321\}.$$

If we compute $X(t)$ term by term using the major index we obtain

$$X(t) = 1 + \quad 1 + t^2 + t + \quad 1 + t^2 + t + \quad 1 + t + t^2 + \\ 1 + t^2 + t + t^2 + t + t^3 = 5 + 5t + 5t^2 + t^3.$$

Taking the cyclic group $C = \mathbb{Z}_3$ acting on the positions $[3]$ to be of order 3, one has

$$X(t) \equiv 6 + 5t + 5t^2 \mod t^3 - 1$$

whose constant term 6 counts the total number of $\mathbb{Z}_3$-orbits, and whose coefficient 5 on t^1 counts the 5 free $\mathbb{Z}_3$-orbits, namely all but the singleton orbit $\{111\}$.

Taking the cyclic group $C = \mathbb{Z}_2$ acting on the first two positions and fixing the third position, one has

$$X(t) \equiv 10 + 6t \mod t^2 - 1$$

whose constant term 10 counts the total number of $\mathbb{Z}_2$-orbits, and whose coefficient of 6 on t^1 counts the 6 free $\mathbb{Z}_2$-orbits, namely those other than the singleton orbits

$$\{111\}, \{112\}, \{113\}, \{221\}.$$

1.4. **Nonnegative matrices.** One can extend the notion of principal specialization to polynomials $f(\mathbf{x}, \mathbf{y})$ in two sets of variables $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_n)$ by defining their *principal (u,t)-specialization*

$$f(1, u, u^2, \ldots, u^{m-1}, 1, t, t^2, \ldots, t^{n-1}).$$

Theorem 4. *Fix a nonnegative integer k, and let X be the collection of all $m \times n$ matrices $A = (a_{ij})$ with entries in the nonnegative integers $\mathbb{N}$ (resp. in $\{0, 1\}$) such that $\sum_{ij} a_{ij} = k$.*

Let $C_1 \times C_2$ act on X, where C_1 and C_2 are cyclic groups acting nearly freely on the row indices $[m]$ and column indices $[n]$. In the case where X consists of $\{0,1\}$-matrices, make the additional assumption that $C_1 \times C_2$ is of **odd** *order.*

Let $X(u, t)$ be the principal (u, t)-specialization of the row-sum and column-sum generating function

$$\sum_{A=(a_{ij}) \in X} \left(\prod_{i,j} (x_i y_j)^{a_{ij}} \right).$$

Then $(X, X(u,t), C_1 \times C_2)$ exhibits the CSP.

Example. Take $k = 2$ and $m = n = 2$, so that X consists of 2×2 nonnegative matrices with entries that sum to 2. Then $C_1, = \langle c_1 \rangle$ and $C_2 = \langle c_2 \rangle$ are cyclic

groups of order two that swap the rows and columns, respectively. The set X decomposes into these four $C_1 \times C_2$-orbits

$$\left\{\begin{matrix} \underset{1}{\begin{bmatrix}2&0\\0&0\end{bmatrix}} & \underset{t^2}{\begin{bmatrix}0&2\\0&0\end{bmatrix}} \\ \\ \underset{u^2}{\begin{bmatrix}0&0\\2&0\end{bmatrix}} & \underset{u^2t^2}{\begin{bmatrix}0&0\\0&2\end{bmatrix}}\end{matrix}\right\}, \left\{\begin{matrix} \underset{t}{\begin{bmatrix}1&1\\0&0\end{bmatrix}} \\ \\ \underset{u^2t}{\begin{bmatrix}0&0\\1&1\end{bmatrix}}\end{matrix}\right\}, \left\{\underset{u}{\begin{bmatrix}1&0\\1&0\end{bmatrix}} \quad \underset{ut^2}{\begin{bmatrix}0&1\\0&1\end{bmatrix}}\right\}, \left\{\underset{ut}{\begin{bmatrix}1&0\\0&1\end{bmatrix}} \quad \underset{ut}{\begin{bmatrix}0&1\\1&0\end{bmatrix}}\right\}$$

in which each element of X is shown with the term it contributes to $X(u,t)$ just below it. The orbits are arranged tabularly so that C_1, C_2 swap bottom-to-top and left-to-right, respectively. The first orbit is $C_1 \times C_2$-free, while in the second orbit c_2 acts trivially, in the third orbit c_1 acts trivially, and in the fourth orbit c_1 and c_2 both swap the two elements of the orbit.

From the data above (or see Section 4 below) one can calculate

$$\begin{aligned} X(u,t) &= (1+u+u^2)(1+t+t^2) + 2ut \\ &\equiv 4 + 2u + 2t + 2ut \mod (u^2-1, t^2-1) \end{aligned}$$

In this last expression, the constant term 4 matches the total number of orbits. One also has

$$X(+1,-1) = X(-1,+1) = X(-1,-1) = 2$$

counting the elements in the second, third, fourth orbits respectively, as they are the elements fixed by $(1,c_2), (c_1,1), (c_1,c_2)$, respectively.

1.5. **Graphs.** One of our main CSP constructions will yield the following result immediately.

Theorem 5. *Let X be any of the following collections of graphs with k edges on vertex set $[n]$:*

(i) *graphs allowing multiedges and loops, including the possibility of multiple loops on the same vertex,*
(ii) *graphs allowing multiedges, but no loops,*
(iii) *graphs allowing no multiedges, and at most one loop on each vertex,*
(iv) *simple graphs, that is, allowing neither multiedges, nor loops.*

Let C be a cyclic group of acting nearly freely on the vertex set $[n]$, and thereby permuting the collection of graphs X. Furthermore, in cases (ii),(iii),(iv) make the additional assumption that C has **odd** *order* [2].

Let $X(u)$ be the u-principal specialization of the degree sequence generating function

$$\sum_{G \in X} \left(\prod_i x_i^{\deg_G(i)} \right) \tag{3}$$

where the vertex-degree $\deg_G(i)$ of vertex i counts edges incident to i with multiplicity, with each loop incident to i contributing 2 to $\deg_G(i)$.

Then $(X, X(u), C)$ exhibits the CSP.

[2]For example, depending upon the odd/even parity of n, one can take C to be generated by an n-cycle/ $(n-1)$-cycle.

Example. Take $n = 3, k = 3$ and consider case (iii) in Theorem 5, so that X consists of graphs on vertex set [3], with 3 edges total, disallowing multiedges, and allowing at most one loop on each vertex. Let C be a cyclic group of order 3 cycling the vertices. Note that C has odd order, as required in case (iii).

One can readily check that there are 20 graphs in X, comprising six free C-orbits, and two singleton orbits: the graph having 3 loops, and the triangle graph having no loops.

One can also calculate that

$$\begin{aligned} X(u) &= 2u^3 + 2u^4 + 4u^5 + 4u^6 + 4u^7 + 2u^8 + 2u^9 \\ &\equiv 8 + 6u + 6u^2 \mod (u^3 - 1) \end{aligned}$$

In this last expression, the constant term 8 counts the total number of orbits, and $X(e^{\frac{2\pi i}{3}}) = 2$ counts the two graphs in the singleton orbits, fixed by C.

2. Constructions and proofs

The representation theoretic paradigm for proving a CSP is based on a simple observation, Proposition 6 below (*cf.* [12, §2]).

Start with the product $\mathbf{C} = C_1 \times \cdots \times C_m$ of finite cyclic groups. After picking embeddings of groups $\omega_i : C_i \hookrightarrow \mathbb{C}^\times$, note that the irreducible representations of $\mathbf{C}$ are exactly the degree one characters $\omega^{\mathbf{d}} := \prod_i \omega_i^{d_i}$ from (1) where $\mathbf{d} = (d_1, \ldots, d_m)$ satisfies $0 \leq d_i < |C_i|$.

Consequently, given any Laurent polynomial in the variable set $\mathbf{u} = (u_1, \ldots, u_d)$

$$X(\mathbf{u}) = \sum_{\mathbf{d} \in \mathbb{Z}^m} a_{\mathbf{d}} \mathbf{u}^{\mathbf{d}} \tag{4}$$

having nonnegative coefficients $a_{\mathbf{d}}$, one can construct a $\mathbf{C}$-representation

$$V_{X(\mathbf{u})} := \bigoplus_{\mathbf{d} \in \mathbb{Z}^m} \left(\omega^{\mathbf{d}}\right)^{\oplus a_{\mathbf{d}}}$$

expressed as an explicit direct sum of the $\mathbf{C}$-irreducibles, with multiplicities.

Given a finite set X permuted by any group G, let $\mathbb{C}[X]$ denote the associated permutation representation of G, having a $\mathbb{C}$-vector space basis indexed by the elements of X.

Proposition 6. *Given a finite set X permuted by a finite product of cyclic groups $\mathbf{C}$, and a Laurent polynomial $X(\mathbf{u})$ having nonnegative coefficients, the triple exhibits the CSP if and only if one has an isomorphism of $\mathbf{C}$-representations*

$$\mathbb{C}[X] \cong V_{X(\mathbf{u})}.$$

Proof. The CSP asserts that every $\mathbf{c}$ in $\mathbf{C}$ has the same character value on $\mathbb{C}[X]$, namely $|X^{\mathbf{c}}|$, as its character value on $V_{X(\mathbf{u})}$, namely $[X(\mathbf{u})]_{u_i = \omega(c_i)}$. $\square$

2.1. **The tensor product construction.**

Proposition 7. *If both $(X_1, X_1(u), C_1)$ and $(X_2, X_2(u), C_2)$ exhibit the CSP, then so does*

$$(X_1 \times X_2, \quad X(u,t) := X_1(u) X_2(t), \quad \mathbf{C} := C_1 \times C_2).$$

Proof. Combine Proposition 6 with the isomorphism of $\mathbf{C}$-representations

$$\mathbb{C}[X_1 \times X_2] \cong \mathbb{C}[X_1] \otimes \mathbb{C}[X_2].$$

$\square$

2.2. **The symmetric and exterior power constructions.** Here we review the notational tool of plethystic composition of symmetric functions; see [15, Chap. 7 Appendix 2].

A *symmetric function* $f(x_1, x_2, \ldots)$ on the infinite variable set $x_1, x_2, \ldots$ with coefficients in a ring R is a power series in $R[x_1, x_2, \ldots]$ of bounded degree which is invariant under all permutations of the subscripts on the variables. One can define the *plethystic composition* $f[X(\mathbf{u})]$ of such a symmetric function f with a Laurent polynomial $X(\mathbf{u})$ as in (4) in the following way: if $n := \sum_{\mathbf{d}} a_{\mathbf{d}} = X(1, \ldots, 1)$, then one substitutes $x_j = 0$ for all $j > n$ in f, and $x_j = u_1^{d_1} \cdots u_m^{d_m}$ for exactly $a_{\mathbf{d}}$ of the variables x_j with $1 \leq j \leq n$. For example, if

$$X(u) = [n]_u = 1 + u + u^2 \cdots + u^{n-1}$$

then $f[X(u)] = f(1, u, u^2, \ldots, u^{n-1})$ is the *principal u-specialization* described earlier.

We also review the meaning of symmetric functions and plethystic composition, with regard to the representations of the general linear group; see [15, Chap. 7 Appendix 2]. Let V be a complex vector space over $\mathbb{C}$ of dimension n, and $GL(V) \cong GL_n(\mathbb{C})$ the general linear group. One says that a representation $GL(V) \stackrel{\rho}{\to} GL(U)$ is *polynomial* if for some (equivalently, any) choice of $\mathbb{C}$-bases for V, U, the matrices representing the action of $\rho(g)$ on U have entries which are polynomial functions in the entries of the matrices representing the action of g on V. A polynomial representation ρ gives rise to a symmetric function $f_\rho(x_1, \ldots, x_n)$ in a finite variable set (with nonnegative integer coefficients) called its *character*, which is the trace of any element g in $GL(V)$ having eigenvalues $x_1, \ldots, x_n$. One can then interpret the plethystic composition $f_\rho[X(\mathbf{u})]$ for a Laurent polynomial $X(\mathbf{u})$ as in (4) as follows: if g is an element of $GL(V)$ whose eigenvalues are given by the $\mathbf{u}^{\mathbf{d}}$ with eigenvalue multiplicity $a_{\mathbf{d}}$, then $\rho(g)$ acts on U with trace $f_\rho[X(\mathbf{u})]$.

The irreducible polynomial representations $GL(V) \stackrel{S^\lambda}{\to} GL(U)$ are indexed by partitions λ having at most $n = \dim V$ parts. The character $f_{S_\lambda}(\mathbf{x})$ is the *Schur function* $s_\lambda(\mathbf{x})$ in variablex $x_1, \ldots, x_n$, which has a well-known expression (see, *e.g.*, [15, §7.10]) as a sum over all *column-strict tableaux* P with entries in $[n] = \{1, 2, \ldots, n\}$:

$$s_\lambda(\mathbf{x}) = \sum_P \mathbf{x}^P \tag{5}$$

where $\mathbf{x}^P := \prod_{i,j} x_{P_{i,j}}$. Two prototypical examples of these polynomial representations, are the k^{th} *symmetric power* $U = \mathrm{Sym}^k(V)$ and the k^{th} *exterior power* $U = \bigwedge^k(V)$, corresponding to the partitions $\lambda = (k)$ and $\lambda = (1^k)$, respectively. Their characters are

$$f_{\mathrm{Sym}^k}(\mathbf{x}) = s_{(k)}(\mathbf{x}) = h_k(\mathbf{x}) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}$$

$$f_{\bigwedge^k}(\mathbf{x}) = s_{(1^k)}(\mathbf{x}) = e_k(\mathbf{x}) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}.$$

We apply these to obtain two more CSP constructions involving sets and multisets. For a finite set X and a nonnegative integer k, let

$$\binom{X}{k} \quad \text{and} \quad \left(\!\!\binom{X}{k}\!\!\right)$$

denote the collection of all k-element subsets and k-element multisubsets of X.

Proposition 8 (*cf.* [12, Theorem 1.1])**.** *If a triple* $(X, X(\mathbf{u}), \mathbf{C})$ *exhibits the CSP, then the triple*

$$\left(\quad \left(\!\!\binom{X}{k}\!\!\right), \quad h_k[X(\mathbf{u})], \quad \mathbf{C} \quad \right)$$

also exhibits the CSP.

If, in addition, $\mathbf{C}$ *has* **odd** *order, then the triple*

$$\left(\quad \binom{X}{k}, \quad e_k[X(\mathbf{u})], \quad \mathbf{C} \quad \right)$$

also exhibits the CSP.

Proof. In either case, Proposition 6 shows that the space $V := \mathbb{C}[X]$ has a $\mathbb{C}$-basis of eigenvectors $\{v_\alpha\}$ diagonalizing the action of $\mathbf{C}$, in such a way that a typical element $\mathbf{c} = (c_1, \ldots, c_m)$ in $\mathbf{C}$ acts with eigenvalue $\left[\mathbf{u}^{\mathbf{d}}\right]_{u_i=\omega_i(c_i)}$ on exactly $a_{\mathbf{d}}$ of the eigenvectors v_α. Thus $\mathbf{c}$ acts on $U = \mathrm{Sym}^k(V)$ and $U = \bigwedge^k(V)$ with traces $h_k[X(\mathbf{u})]_{u_i=\omega_i(c_i)}$ and $e_k[X(\mathbf{u})]_{u_i=\omega_i(c_i)}$, respectively.

On the other hand, we claim that one has isomorphisms of $\mathbf{C}$-representations

$$\mathrm{Sym}^k(V) \cong \mathbb{C}\left[\left(\!\!\binom{X}{k}\!\!\right)\right]$$
$$\bigwedge{}^k(V) \cong \mathbb{C}\left[\binom{X}{k}\right]$$

To see this, note $V = \mathbb{C}[X]$ has a $\mathbb{C}$-basis $\{e_x\}_{x\in X}$ permuted in the same way that $\mathbf{C}$ acts on X. Therefore $\mathrm{Sym}^k(V)$ and $\bigwedge^k(V)$ have $\mathbb{C}$-bases of monomial symmetric tensors $e_{x_1} \cdots e_{x_k}$ and antisymmetric tensors $e_{x_1} \wedge \cdots \wedge e_{x_k}$. The group $\mathbf{C}$ permutes symmetric tensors $e_{x_1} \cdots e_{x_k}$ in exactly the way it permutes k-element multisets. When the group $\mathbf{C}$ acts on antisymmetric tensors, it does not quite act on them in the way that it permutes k-element subsets, but rather permutes and scales them by a sign of ± 1. However, here one uses the assumption that $\mathbf{C}$ has odd order: when some $\mathbf{c}$ in $\mathbf{C}$ *fixes* some $e_{x_1} \wedge \cdots \wedge e_{x_k}$ up to sign, meaning that

$$\mathbf{c}(e_{x_1} \wedge \cdots \wedge e_{x_k}) = \pm e_{x_1} \wedge \cdots \wedge e_{x_k},$$

then $\mathbf{c}$ having odd order forces this sign to be $+1$. Thus each $\mathbf{c}$ in $\mathbf{C}$ acts with the same trace in $\bigwedge^k(V)$ as in $\mathbb{C}\left[\binom{X}{k}\right]$. ☐

Proof of Theorem 5. We will prove a stronger statement. Assume one has a triple $(X, X(\mathbf{u}), \mathbf{C})$ exhibiting the CSP. Proposition 8 then shows that the triples

$$\left(\left(\!\!\binom{X}{m}\!\!\right), \quad h_m\left[X(\mathbf{u})\right], \quad \mathbf{C} \quad \right)$$
$$\left(\binom{X}{m}, \quad e_m\left[X(\mathbf{u})\right], \quad \mathbf{C} \quad \right)$$

both also exhibit the CSP, assuming that $\mathbf{C}$ is of odd order in the latter case. Applying Proposition 8 one more time then shows the following result.

Theorem 9. *Given a triple* $(X, X(\mathbf{u}), \mathbf{C})$ *exhibiting the CSP, the following triples also exhibit the CSP*

(i)

$$\left(\left(\!\!\left(\left(\!\!\left(\begin{matrix} X \\ m \end{matrix}\right)\!\!\right) \atop k\right)\!\!\right), \quad h_k\left[h_m\left[X(\mathbf{u})\right]\right], \quad \mathbf{C} \right)$$

(ii)

$$\left(\left(\!\!\left(\left(\begin{matrix} X \\ m \end{matrix}\right) \atop k\right)\!\!\right), \quad h_k\left[e_m\left[X(\mathbf{u})\right]\right], \quad \mathbf{C} \right)$$

(iii)

$$\left(\left(\left(\!\!\left(\begin{matrix} X \\ m \end{matrix}\right)\!\!\right) \atop k\right), \quad e_k\left[h_m\left[X(\mathbf{u})\right]\right], \quad \mathbf{C} \right)$$

(iv)

$$\left(\left(\left(\begin{matrix} X \\ m \end{matrix}\right) \atop k\right), \quad e_k\left[e_m\left[X(\mathbf{u})\right]\right], \quad \mathbf{C} \right)$$

under the extra assumption that $\mathbf{C}$ *is of* **odd** *order in cases (ii),(iii),(iv).*

A little reflection shows that Theorem 5 is the special case of Theorem 9 in which one takes $m = 2$, with $X = [n]$ permuted nearly freely by a cyclic group C, and $X(\mathbf{u}) = [n]_u$. $\square$

Remark 10. The assumption that $|C|$ is odd in cases (ii),(iii),(iv) of Theorem 5 is perhaps too restrictive; we have not made an exhaustive study of the exact hypotheses on $|C|$ and k which are necessary and sufficient for these triples to exhibit the CSP. However, we do offer one instance of a negative result in this regard, as an indication of what one might expect (*cf.* [12, Lemma 2.3]).

Proposition 11. *Let X be the class of graphs with k edges satisfying the conditions of Theorem 5(iii) or (iv), and let $X(u)$ be as defined there, as the u-principal specialization of* (3).

Let $C = \langle c \rangle = \mathbb{Z}_n$ with $c = (1, 2, \ldots, n)$ cyclically permuting the vertex set $[n]$, and assume that n is **even**.

Then the triple $(X, X(u), C)$ exhibits the CSP if and only if either

- $k \in \{0, 1, \binom{n}{2}, \binom{n}{2} - 1\}$.
- $n = 4, k = 3$.

Proof. The "if" direction can be trivially verified. For the "only if" direction, we prove something stronger: For even $n \geq 6$ and $2 \leq k \leq \binom{n}{2} - 2$, *there is no integer m such that $u^m X(u)$ gives a triple $(X, u^m X(u), C)$ that exhibits the CSP.*

If there were such an integer m, then for each c^d in C and $\omega = e^{\frac{2\pi i}{n}}$ one would have

$$|X^{c^d}| = |\omega^{dm} X(\omega^d)| = |X(\omega^d)|.$$

The proof of Theorem 9 shows that $X(\omega^d)$ is the trace of c^d acting on $U = \mathrm{Sym}^k(\bigwedge^2(V))$ or $U = \bigwedge^k(\bigwedge^2(V))$, in either case. Since these spaces U have $\mathbb{C}$-bases

of monomial tensors v_G indexed by graphs G, permuted by C up to root-of-unity scalar multiples, one has

$$|X(\omega^d)| = \text{ trace of } c^d \text{ on } U = \left| \sum_{\substack{G\in X:\\ c^d(G)=G}} \frac{c^d(v_G)}{v_G} \right| \leq \sum_{\substack{G\in X:\\ c^d(G)=G}} \left| \frac{c^d(v_G)}{v_G} \right| = |X^{c^d}|$$

since each of the scalars $\frac{c^d(v_G)}{v_G}$ is a root-of-unity, with complex modulus 1. Furthermore, the case of equality occurs if and only if these scalars $\frac{c^d(v_G)}{v_G}$ for graphs G fixed by c^d are all equal, independent of the choice of G. Thus one can disprove the existence of such a CSP by exhibiting a choice of d and a choice of two graphs G, G' fixed by c^d such that $\frac{c^d(v_G)}{v_G} \neq \frac{c^d(v_{G'})}{v_{G'}}$.

When $n = 2m$ is even, and $2 \leq k \leq \binom{n}{2} - 2$ and $n \geq 6$, one can exhibit such G, G' fixed by $c^m = (1, m+1)(2, m+2)(3, m+3)\cdots(m-1, n-1)(m, n)$. Create G by starting with the two edges $\{1, m+1\}, \{2, m+2\}$ and completing G with any $k-2$ other edges that make $c^m(G) = G$. Then obtain G' from G by replacing the above two edges with $\{1, m+2\}, \{2, m+1\}$, and leaving the other $k-2$ edges of G the same. One then calculates in $U = \bigwedge^k(\bigwedge^2(V))$ or $U = \bigwedge^k(\mathrm{Sym}^2(V))$ that

$$\begin{aligned} c^m\left((e_1 \wedge e_{m+1}) \wedge (e_2 \wedge e_{m+2})\right) &= (e_{m+1} \wedge e_1) \wedge (e_{m+2} \wedge e_2) \\ &= +(e_1 \wedge e_{m+1}) \wedge (e_2 \wedge e_{m+2}) \\ \\ c^m\left((e_1 \cdot e_{m+1}) \wedge (e_2 \cdot e_{m+2})\right) &= (e_{m+1} \cdot e_1) \wedge (e_{m+2} \cdot e_2) \\ &= +(e_1 \cdot e_{m+1}) \wedge (e_2 \cdot e_{m+2}), \end{aligned}$$

while,

$$\begin{aligned} c^m\left((e_1 \wedge e_{m+2}) \wedge (e_2 \wedge e_{m+1})\right) &= (e_{m+1} \wedge e_2) \wedge (e_{m+2} \wedge e_1) \\ &= -(e_1 \wedge e_{m+2}) \wedge (e_2 \wedge e_{m+1}) \\ \\ c^m\left((e_1 \cdot e_{m+2}) \wedge (e_2 \cdot e_{m+1})\right) &= (e_{m+1} \cdot e_2) \wedge (e_{m+2} \cdot e_1) \\ &= -(e_1 \cdot e_{m+2}) \wedge (e_2 \cdot e_{m+1}) \end{aligned}$$

Since G, G' share all $k-2$ other edges, we have

$$\frac{c^m(v_G)}{v_G} = -\frac{c^m(v_{G'})}{v_{G'}},$$

in both cases. $\square$

Proof of Theorem 4. Let C_1, C_2 act nearly freely on $[m], [n]$ respectively, so that the triples $([m], [m]_u, C_1)$ and $([n], [n]_t, C_2)$ both exhibit the CSP by Proposition 15. Proposition 7 implies

$$(\quad [m] \times [n], \quad [m]_u [n]_t, \quad C_1 \times C_2 \quad)$$

also exhibits the CSP, and then Proposition 8 shows that

$$\left(\quad \left(\!\!\binom{[m] \times [n]}{k}\!\!\right), \quad h_k\left[\ [m]_u[n]_t\ \right], \quad C_1 \times C_2 \quad \right)$$
$$\left(\quad \binom{[m] \times [n]}{k}, \quad e_k\left[\ [m]_u[n]_t\ \right], \quad C_1 \times C_2 \quad \right)$$

also exhibit the CSP, assuming that $C_1 \times C_2$ is of odd order in the latter case.

Now use the usual bijection between k-element subsets (resp., multisubsets) of $[m] \times [n]$ and $m \times n$ matrices $A = (a_{ij})$ whose entries sum to k having $\{0,1\}$ (resp., nonnegative integer) entries: the entry a_{ij} gives the multiplicity with which the element (i,j) of $[m] \times [n]$ appears in the k-element subset (resp., multiset). $\square$

Remark 12. It is perhaps worth comparing Theorem 4 with recent results of Rhoades [11]. He again considers a subset X of all matrices $A = (a_{ij})$ having nonnegative (resp., $\{0,1\}$) entries. However his matrices are defined by having fixed row and column sum vectors μ, ν, such that μ, ν are invariant under cyclic groups C_1, C_2. Thus the product $C_1 \times C_2$ again acts on X by having $C_1 \times C_2$ permute row, column indices. His results [11, Theorems 1.3, 1.4] describe generating functions $X(u,t)$ for a triple $(X, X(u,t), C_1 \times C_2)$ exhibiting a CSP in this situation, derived from Kostka-Foulkes polynomials, and related to the charge statistics on biwords.

2.3. **The tensor power construction.** Our last construction makes use of two basic representation theoretic facts: Schur-Weyl duality in tensor powers $V^{\otimes \ell}$ and the type A case of Springer's theory of regular elements. We quickly review these here.

Proposition 13. *(Schur-Weyl duality) Regarding the ℓ-fold tensor product $V^{\otimes \ell}$ as a $GL(V) \times \mathfrak{S}_\ell$-representation in which $GL(V)$ acts diagonally and $\mathfrak{S}_\ell$ permutes the tensor positions $[\ell]$, one has the following irreducible decomposition:*

$$V^{\otimes \ell} \cong \bigoplus_{\lambda \vdash \ell} S^\lambda \otimes \chi^\lambda, \tag{6}$$

where S^λ, χ^λ, respectively, are the irreducible representations of $GL(V), \mathfrak{S}_\ell$, respectively, indexed by λ.

Springer [13] introduced the following crucial notion.

Definition. A *regular element* in a finite subgroup $W \subset GL(U)$ generated by (complex) reflections is defined to be an element c that has an eigenvector lying in U^{reg}, where U^{reg} is the complement within U of the reflecting hyperplanes for the elements of W.

Given an irreducible W-character χ, the value $\chi(c)$ on a regular element turns out to be determined by the *fake-degree polynomial* $f^\chi(t)$, defined as the polynomial whose coefficient of t^d gives the multiplicity of χ within the d^{th} graded component of the *coinvariant algebra* $\mathbb{C}[U]/(\mathbb{C}[U]^W_+)$; see [13, §2.5.].

Theorem 14. [13, Proposition 4.5] *Let c be a regular element c in a finite complex reflection group W acting on U, say with eigenvalue ω on some eigenvector in U^{reg}. Then for any W-irreducible character χ, one has $\chi(c) = f^\chi(\omega^{-1})$*

Example. Regarding $W = \mathfrak{S}_\ell$ as a complex reflection group acting on $U = \mathbb{C}^\ell$ by permuting coordinates, there is a formula (due originally to Lusztig; see [16, Prop. 4.11]) for the fake-degree polynomials $f^\lambda(t)$ associated to the irreducible χ^λ, as a sum over *standard Young tableaux* Q of shape λ

$$f^\lambda(t) = \sum_Q t^{\mathrm{maj}(Q)} \tag{7}$$

in which $\mathrm{maj}(Q)$ is the sum of those entries i in Q for which $i+1$ appears in a lower row than i. One can also readily check the following:

Proposition 15 (*cf.* [13, §5.1]). *Let $W = \mathfrak{S}_\ell$ regarded as a complex reflection group acting on $U = \mathbb{C}^\ell$ by permuting coordinates. The following are equivalent for an element c in $\mathfrak{S}_\ell$*

(i) *c is regular.*
(ii) *c permutes the set of coordinates $[\ell]$ nearly freely.*
(iii) *If c has multiplicative order d, then every primitive d^{th} root of unity is achieved as an eigenvalue for c on at least one eigenvector lying in U^{reg}.*
(iv) *The triple*

$$(X := [\ell], \quad X(u) := [\ell]_u, \quad C := \langle c \rangle)$$

exhibits the CSP.

In particular, for any embedding $C \xrightarrow{\omega} \mathbb{C}^\times$ of the cyclic group $C = \langle c \rangle$, such elements c as in (i)-(iv) satisfy $\chi^\lambda(c) = \left[f^\lambda(t)\right]_{t=\omega(c)}$.

Proposition 16. *Let $(X, X(\mathbf{u}), \mathbf{C})$ be a triple that exhibits the CSP, and let C be a cyclic group permuting $[\ell] = \{1, 2, \ldots, \ell\}$ nearly freely. Let X^ℓ be the collection of words of length ℓ in the alphabet, permuted by $\mathbf{C} \times C$ in which $\mathbf{C}$ acts on the letter values, and C acts on the positions $[\ell]$.*

Then

$$\left(\quad X^\ell, \quad f[X(\mathbf{u})](t), \quad \mathbf{C} \times C \quad \right)$$

exhibits the CSP, where $f(\mathbf{x})(t) := f(\mathbf{x}, t)$ is the symmetric function with cofficients in $\mathbb{Z}[t]$ appearing in (2).

Proof. Let $V = \mathbb{C}[X]$, and let $n := |X| = \dim_{\mathbb{C}} V$. One has an isomorphism of $\mathbf{C} \times C$-representations $\mathbb{C}[X^\ell] \cong V^{\otimes \ell}$, in which $\mathbf{C}$ inherits its action by restriction from the diagonal $GL(V)$-action on $V^{\otimes \ell}$, and C inherits its action by restriction from the $\mathfrak{S}_\ell$ permuting the tensor positions in $V^{\otimes \ell}$. Thus Schur-Weyl duality (6) implies that for any element $\mathbf{x}$ in $GL(V)$ having eigenvalues $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and any element c in C, the trace of $(\mathbf{x}, c)$ acting on $V^{\otimes \ell}$ will be

$$\sum_{\lambda \vdash \ell} s_\lambda(\mathbf{x}) \chi^\lambda(c). \tag{8}$$

Using Theorem 14 and the tableaux expressions (5),(7) for $s_\lambda(\mathbf{x})$ and for $f^\lambda(t)$, one can rewrite (8) as

$$\left[\sum_{(P,Q)} \mathbf{x}^P t^{\mathrm{maj}(Q)} \right]_{t=\omega(c)} \tag{9}$$

in which (P, Q) run through all pairs of Young tableaux of the same shape, with P column-strict and Q standard. Well-known properties of the Robinson-Schensted-Knuth bijection [15, SS7.11, 7.23] then let one rewrite this trace of $(\mathbf{x}, c)$ as

$$\left[\sum_{w \in X^\ell} \mathbf{x}^w t^{\mathrm{maj}(w)} \right]_{t=\omega(c)} = [f(\mathbf{x}, t)]_{t=\omega(c)}.$$

Given an element $\mathbf{c} = (c_1, \ldots, c_m)$ in $\mathbf{C}$, when it is considered as an element of $GL(V)$, it has exactly $a_{\mathbf{d}}$ of its eigenvalues equal to $\prod_i \omega_i(c_i)$. Hence the discussion of plethysm in Subsection 2.2 shows the trace of $(\mathbf{c}, c)$ on $V^{\otimes \ell}$ will be $[f(\mathbf{x}, t)[X(\mathbf{u})]]_{u_i=\omega_i(c_i), t=\omega(c)}$ as desired. □

Proof of Theorem 1. This is immediate from Propositions 15 and 16. □

Proof of Theorem 2. As in the statement of the theorem, consider $X = \mathbb{F}_{q^\ell}$ with action of $C_1 \times C_2$ where $C_1 = \mathbb{F}_q^\times$ acts by scalar multiplication and $C_2 = \mathrm{Gal}(\mathbb{F}_{q^\ell}/\mathbb{F}_q)$ acts by powers of the Frobenius endomorphism F.

Since $\mathbb{F}_{q^\ell}/\mathbb{F}_q$ is a Galois extension, the Normal Basis Theorem (see, *e.g.*, Lang [7, Chap. VIII Theorem 13.1]) implies that there exists an element $\alpha \in \mathbb{F}_n$ whose Galois images $\{\alpha, F(\alpha), F^2(\alpha), \ldots, F^{\ell-1}(\alpha)\}$ give an $\mathbb{F}_q$-basis for $\mathbb{F}_{q^\ell}$. This choice of basis gives an $\mathbb{F}_q$-vector space isomorphism $\mathbb{F}_q^\ell \to \mathbb{F}_{q^\ell}$. Taking $n = q$, one can precompose this with a bijection $[n]^\ell \to \mathbb{F}_q^\ell$ that comes from numbering the elements of $\mathbb{F}_q$ by $[n]$. The composite is a bijection $[n]^\ell \to \mathbb{F}_{q^\ell}$ which is $C_1 \times C_2$-equivariant, where the C_1-action on the letter values $[n]$ is nearly free, fixing only the value that labels the zero element of $\mathbb{F}_q$, and the C_2-action freely permutes the positions in the words $[n]^\ell$ cyclically. □

3. Parking functions

We prove here something somewhat more general than Theorem 3, and then remark on the relation to Theorem 3.

Proposition 17. *Let X by any collection of words in $[n]^\ell$ which is stable under the action of $\mathfrak{S}_\ell$ permuting positions, and let C be a cyclic subgroup of $\mathfrak{S}_\ell$ permuting the positions $[\ell]$ nearly freely.*

Let $X(t) = \sum_{w \in X} t^{\mathrm{maj}(w)}$.

Then the triple $(X, X(t), C)$ exhibits the CSP.

Proof. It suffices to prove this in the special case where X is the $\mathfrak{S}_\ell$-orbit of one word w. If w has k_i occurrences of the letter i, then we claim that the $\mathfrak{S}_\ell$-action on X is isomorphic to the $\mathfrak{S}_\ell$-action on flags of nested subsets

$$\emptyset \subset S_{k_1} \subset S_{k_1+k_2} \subset S_{k_1+k_2+k_3} \subset \cdots \subset [\ell]$$

having cardinalities $k_1, k_1 + k_2, k_1 + k_2 + k_3, \ldots$. This follows because both such $\mathfrak{S}_\ell$-actions are transitive, and have the stabilizer of a typical element conjugate to the Young subgroup $\mathfrak{S}_{k_1} \times \mathfrak{S}_{k_2} \times \cdots \times \mathfrak{S}_{k_n}$.

Hence [12, Proposition 4.4] says that one has a CSP triple $(X, X(q), C)$ where

$$X(q) = \begin{bmatrix} \ell \\ k_1, \ldots, k_n \end{bmatrix}_q$$

is the *q-multinomial coefficient*. On the other hand, MacMahon showed that

$$\begin{bmatrix} \ell \\ m_1, \ldots, m_n \end{bmatrix}_q = \sum_{w \in X} q^{\mathrm{maj}(w)}.$$ □

Remark 18. The case of Theorem 3 in which the cyclic group C permutes the parking functions nearly freely while fixing the ℓ^{th} coordinate, so that $C \subset \mathfrak{S}_{\ell-1}$, also follows from Theorem 1 by the following reasoning.

Let $\mathbb{Z}_{\ell+1}$ denote the integers mod $\ell+1$. Consider its ℓ-fold Cartesian product $\mathbb{Z}_{\ell+1}^{\ell}$ with $\mathfrak{S}_\ell$ acting by permuting positions. This descends to an action of $\mathfrak{S}_\ell$ on the quotient group $\mathbb{Z}_{\ell+1}^{\ell}/\mathbb{Z}\mathbf{1}$ where $\mathbb{Z}\mathbf{1}$ is the diagonal subgroup generated by $\mathbf{1} := (1,1,\ldots,1)$.

There are two well-known collections of coset representatives for this quotient group:

- The subgroup isomorphic to $\mathbb{Z}_{\ell+1}^{\ell-1}$ consisting of those elements of $\mathbb{Z}_{\ell+1}^{\ell}$ having a zero in the ℓ^{th}-coordinate.
  This gives an $\mathfrak{S}_{\ell-1}$-equivariant bijection $\mathbb{Z}_{\ell+1}^{\ell}/\mathbb{Z}\mathbf{1} \leftrightarrow \mathbb{Z}_{\ell+1}^{\ell-1}$.
- The set P_ℓ of all parking functions of length ℓ; see Haiman [5, Proposition 2.6.1].
  This gives an $\mathfrak{S}_\ell$-equivariant bijection $\mathbb{Z}_{\ell+1}^{\ell}/\mathbb{Z}\mathbf{1} \leftrightarrow P_\ell$.

Composing these two bijections gives an $\mathfrak{S}_{\ell-1}$-equivariant bijection $P_\ell \leftrightarrow \mathbb{Z}_{\ell+1}^{\ell-1}$, and hence also a C-equivariant bijection between these sets.

Therefore ignoring the action on the values in Theorem 1 gives this special case of Theorem 3.

Remark 19. Kung, Sun and Yan [6] discuss generalizations of parking functions, parametrized by the choice of two non-crossing lattice paths. By an appropriate choice of the lattices paths, the type $A_{\ell-1}$ parking functions P_ℓ discussed above, the type B_ℓ parking functions of Biane [3] and Stanley [17], and their "Fuss" generalizations [1] are seen to be special cases of these parking functions. For every choice of non-crossings lattice paths, the associated parking functions are again collections of words which are stable under the action of the symmetric group by permuting positions. It follows that Proposition 17 applies to each of these collections.

4. Hook-length and hook-content formulas

Many of our CSP theorems have expressed the generating functions as $X(\mathbf{u}) = \sum_{x\in X} \mathbf{u}^{\mathbf{s}(x)}$ for some statistic(s) $\mathbf{s}(x)$ on the set X. We point out here how in most of these results, there is a more compact expression for $X(\mathbf{u})$, because it is the principal specialization of a symmetric function having an *explicit* expansions in terms of Schur functions $s_\lambda(\mathbf{x})$, or a Schur function multiplied by fake-degree polynomials $f^\lambda(t)$. The latter objects are expressed as convenient products by the hook-content formula [15, Section 7.21] for principally specialized Schur functions, and the hook formula for the fake-degree polynomials:

$$\begin{aligned} s_\lambda(1,u,u^2,\ldots,u^{n-1}) &= s_\lambda[\ [n]_u\ ] \\ &= u^{b(\lambda)} \prod_{x\in\lambda} \frac{1-u^{n+c(x)}}{(1-u^{h(x)})}, \\ f^\lambda(t) &= t^{b(\lambda)} \frac{(t;t)_k}{\prod_{x\in\lambda}(1-t^{h(x)})}, \end{aligned}$$

where x runs through each of the $k=|\lambda|$ cells of λ in each product, the hooklength $h(x)$ is the number of cells weakly to the right of x plus the number of cells strictly

below it, the content $c(x)$ is $j-i$ if x lies in row i and column j,

$$b(\lambda) := \sum_i (i-1)\lambda_i = \sum_j \binom{\lambda'_j}{2}$$
$$(z;t)_k := (1-z)(1-zt)(1-zt^2)\cdots(1-zt^{k-1}).$$

4.1. **Matrices.** The Cauchy and dual Cauchy identities [15, Theorems 7.12.1 and 7.14.3] assert that the generating function for $m\times n$ nonnegative matrices $A=(a_{ij})$ having entries that sum to k

$$\sum_{A=(a_{ij})\in X} \left(\prod_{i,j}(x_iy_j)^{a_{ij}}\right) = \sum_{\lambda\vdash k} s_\lambda(\mathbf{x})s_\lambda(\mathbf{y}),$$

while for $\{0,1\}$-matrices $A=(a_{ij})$ having entries that sum to k

$$\sum_{A=(a_{ij})\in X} \left(\prod_{i,j}(x_iy_j)^{a_{ij}}\right) = \sum_{\lambda\vdash k} s_\lambda(\mathbf{x})s_{\lambda'}(\mathbf{y}),$$

where λ' denotes the conjugate or transpose partition to λ. Consequently, the two generating functions $X(u,t)$ appearing in Theorem 4 have these more compact expressions:

$$\sum_{\lambda\vdash k}(ut)^{b(\lambda)}\prod_{x\in\lambda}\frac{(1-u^{m+c(x)})(1-t^{n+c(x)})}{(1-u^{h(x)})(1-t^{h(x)})},$$
$$\sum_{\lambda\vdash k}(ut)^{b(\lambda)}\prod_{x\in\lambda}\frac{(1-u^{m+c(x)})(1-t^{n-c(x)})}{(1-u^{h(x)})(1-t^{h(x)})}.$$

4.2. **Words.** The proof of Proposition 16 shows that $X_{n,\ell}(u,t) := X(u,t)$ appearing in Theorem 1 on words $[n]^\ell$ is the principal u-specialization of the variables $\mathbf{x}=(x_1,\ldots,x_n)$ in

$$\sum_{\lambda\vdash\ell} s_\lambda(\mathbf{x})f^\lambda(t)$$

and hence has the more compact expression

$$X_{n,\ell}(u,t) = (t;t)_\ell \sum_{\lambda\vdash\ell}(ut)^{b(\lambda)}\prod_{x\in\lambda}\frac{(1-u^{n+c(x)})}{(1-u^{h(x)})(1-t^{h(x)})}. \tag{10}$$

We remark on how this implies an interesting reciprocity property of these polynomials when regarded as functions of n.

Proposition 20. *We have,*

$$t^{\binom{\ell}{2}}X_{n,\ell}(u,t^{-1}) = (-u^n)^\ell\left[X_{n,\ell}(u,t)\right]_{n\mapsto -n}$$

Proof. Define

$$T_{\lambda,\ell}(n,u,t) := (t;t)_\ell(ut)^{b(\lambda)}\prod_{x\in\lambda}\frac{1-u^{n+c(x)}}{(1-u^{h(x)})(1-t^{h(x)})},$$

so that $X_{n,\ell}(u,t)=\sum_{\lambda\vdash\ell}T_{\lambda,\ell}(n,u,t)$. One checks that

$$t^{\binom{\ell}{2}}T_{\lambda,\ell}(n,u,t^{-1}) = (-u^n)^\ell T_{\lambda',\ell}(-n,u,t)$$

due to the following facts:

- $b(\lambda') - b(\lambda) = \sum_{x\in\lambda} c(x)$,
- $b(\lambda') + b(\lambda) + |\lambda| = \sum_{x\in\lambda} h(x)$,
- λ, λ' share the same hook lengths, and
- the cells x and x' that correspond under conjugation will have opposite contents: $c(x) = -c(x')$. $\square$

4.3. **Graphs.** Each of the polynomials $X(u)$ appearing in Theorem 9(i)-(iv) is a specialization of a plethystic composition of symmetric functions of the form $h_k[h_m], h_k[e_m], e_k[h_m], e_k[e_m]$. Thus whenever one knows their explicit expansion into Schur functions, the principal specialization has a compact expression.

In fact, such plethysm expansions are known when $m = 2$ by various formulas of Littlewood [8, Chap I, §8, Exer. 6], covering all the cases that appear in Theorem 5(i)-(iv). Similarly one has such plethysm expansions whenever $k = 2$ (see [8, Chap I, §8, Exer. 9]).

Appendix A. On the well-definition of the CSP

The data implicit in a CSP is more than just a triple $(X, X(\mathbf{u}), \mathbf{C})$ of a finite set X with the permutation action of a finite abelian group $\mathbf{C}$, and a polynomial $X(\mathbf{u}) := X(u_1, \ldots, u_m)$ in $\mathbb{Z}[\mathbf{u}]$. Implicitly, one must also choose two things:

(a) the decomposition $\mathbf{C} := C_1 \times \cdots \times C_m$, and
(b) the embeddings of groups $\omega_i : C_i \hookrightarrow \mathbb{C}^\times$ used to phrase the CSP assertion that $|X^{\mathbf{c}}| = [X(\mathbf{u})]_{u_i=\omega_i(c_i)}$.

When $m = 1$, so that $\mathbf{C}$ is a single cyclic group $C = \mathbb{Z}_n$, then it is easy to see that $(X, X(u), C)$ exhibits the CSP for some embedding $\omega : C \to \mathbb{C}^\times$ if and only if it exhibits the CSP for any other embedding of C into $\mathbb{C}^\times$. This is because there is always an element of the Galois group $\mathrm{Gal}(\mathbb{Q}[e^{\frac{2\pi i}{n}}]/\mathbb{Q})$ that takes one such embedding to another, fixing the polynomial $X(u)$.

On the other hand, the following example shows that for non-cyclic abelian groups $\mathbf{C}$ the embeddings in (b) can make some difference.

Example 21. Let $X = \mathbb{Z}_3 = \{1, \omega, \omega^{-1}\} \subset \mathbb{C}^\times$ where $\omega := e^{\frac{2\pi i}{3}}$, and let

$$\mathbf{C} = C_1 \times C_2 = \mathbb{Z}_3 \times \mathbb{Z}_3 \subset \mathbb{C}^\times \times \mathbb{C}^\times$$

act on X via $(\alpha, \beta)\cdot\gamma = \alpha\beta\gamma$ where here α, β, γ are all considered inside $\mathbb{C}^\times$. Then one can check that with respect to the natural inclusions ω_1, ω_2 of C_1, C_2 into $\mathbb{C}^\times$, the polynomial $X(u,t) = 1 + ut + u^2t^2$ gives a CSP triple $(X, X(u,t), \mathbf{C})$. However, if one alters the embedding of C_2 so as to send $\beta \mapsto \beta^{-1}$, this is no longer a CSP triple. On the other hand, it can be fixed if one replaces the polynomial $X(u,t)$ with the polynomial $X(u, t^2)$.

Example 22. Let C be a cyclic group such that $C = C_1 \times C_2$, where, necessarily, C_1 and C_2 have relatively prime orders. Suppose that $(X, X(u), C)$ exhibits the CSP (for some and, hence, any embedding $C \to \mathbb{C}^\times$). It follows that $(X, X(u,t), C_1 \times C_2)$ exhibits the CSP, where $X(u,t) = X(ut)$. Indeed, if $\omega_i : C_i \to \mathbb{C}^\times$ are injections for $i = 1, 2$ then $\omega_1\omega_2 : C \to \mathbb{C}^\times$ is an injection. It follows that

$$X(\omega_1(c_1), \omega_2(c_2)) = X(\omega_1(c_1)\omega_2(c_2)) = |X^{(c_1,c_2)}|,$$

which is to say that $(X, X(u,t), C_1 \times C_2)$ exhibits the CSP.

These two examples suggest how one can account for both choices (a) and (b) in general, by altering the polynomial $X(\mathbf{u})$. Suppose one is given two decompositions

$$\mathbf{C} := C_1 \times \cdots \times C_m = C'_1 \times \cdots \times C'_{m'}$$

and accompanying embeddings

$$\omega_i : C_i \hookrightarrow \mathbb{C}^\times, \qquad i = 1, 2, \ldots, m,$$
$$\omega'_j : C'_j \hookrightarrow \mathbb{C}^\times, \qquad j = 1, 2, \ldots, m'.$$

As mentioned in Section 2, every degree one character

$$C_1 \times \cdots \times C_m \to \mathbb{C}^\times$$

can be expressed uniquely in the form $\omega^{\mathbf{d}} = \omega_1^{d_1} \cdots \omega_m^{d_m}$ with $0 \le d_i < |C_i|$ for $i = 1, 2, \ldots, m$. It follows that the composite characters defined for $1 \le j \le m'$ by

$$C_1 \times \cdots \times C_m = \mathbf{C} = C'_1 \times \cdots \times C'_{m'} \xrightarrow{\pi'_j} C'_j \xrightarrow{\omega'_j} \mathbb{C}^\times,$$

where π'_j is the projection map, can each be written as $\omega^{\mathbf{d}^{(j)}}$ for some $\mathbf{d}^{(j)}$. Given variables $\mathbf{u} = (u_1, \ldots, u_m)$ we set

$$\mathbf{u}^{\mathbf{d}^{(j)}} = u_1^{d_1^{(j)}} u_2^{d_2^{(j)}} \cdots u_m^{d_m^{(j)}}.$$

One can now check that

$$(X,\ X(v_1, v_2, \ldots, v_{m'}),\ C'_1 \times \cdots \times C'_{m'})$$

exhibits the CSP, with respect to the embeddings $\{\omega'_j\}_{j=1,2,\ldots,m'}$ if and only if

$$(X,\ X(\mathbf{u}^{\mathbf{d}^{(1)}}, \mathbf{u}^{\mathbf{d}^{(2)}}, \ldots, \mathbf{u}^{\mathbf{d}^{(m')}}),\ C_1 \times \cdots \times C_m)$$

exhibits the CSP with respect to to the embeddings $\{\omega_i\}_{i=1,2,\ldots,m}$.

Reversing the roles of $\{\omega_i\}_{i=1,2,\ldots,m}$ and $\{\omega'_i\}_{i=1,2,\ldots,m'}$, one sees that one can pass between the polynomials relevant for any two CSPs for X and $\mathbf{C}$ by a simple monomial change-of-variables.

Acknowledgements

The authors thank Dennis Stanton for helpful conversations regarding this material. They also thank the referees for a careful reading of the manuscript.

References

[1] D. Armstrong, S.-P. Eu, Nonhomogeneous parking functions and noncrossing partitions, *Electron. J. Combin.* **15** (2008), #R146

[2] H. Barcelo, V. Reiner and D. Stanton, Bimahonian distributions. *J. London Math. Soc.* **77** (2008), 627-646.

[3] P. Biane, Parking functions of types A and B, *Electron. J. Combin.* **9(1)** (2002), N7.

[4] D. Foata and M.-P. Schützenberger, Major index and inversion number of permutations. *Math. Nachr.* **83** (1978), 143–159.

[5] M. D. Haiman, Conjectures on the quotient ring by diagonal invariants. *J. Algebraic Combin.* **3** (1994), no. 1, 17–76.

[6] J. P. S. Kung, X. Sun, and C. Yan, Two-boundary lattice paths and parking functions, *Adv. in Appl. Math.* **39** (2007).

[7] S. Lang, Algebra, *Graduate Texts in Mathematics* **211**. Springer-Verlag, New York, 2002.

[8] I. G. Macdonald, Symmetric functions and Hall polynomials. Second edition. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[9] H. Morita and T. Nakajima, A formula of Lascoux-Leclerc-Thibon and representations of the symmetric groups. *J. Algebraic Combin.* **24** (2006), 45-60.
[10] I. Pak, A. Postnikov, Enumeration of trees and one amazing representation of S_n (extended abstract). Proceedings of FPSAC'96.
[11] B. Rhoades, Hall-Littlewood polynomials and fixed point enumeration. *Disc. Math.* **310** (2010), 869-876.
[12] V. Reiner, D. Stanton, and D. White, The cyclic sieving phenomenon. *J. Combin. Theory Ser. A* **108** (2004), 17–50.
[13] T. A. Springer, Regular elements of finite reflection groups. *Invent. Math.* **25** (1974), 159–198.
[14] R. P. Stanley, Enumerative Combinatorics, Volume 1. *Cambridge Studies in Advanced Mathematics* **49**. Cambridge University Press, Cambridge, 1997.
[15] R. P. Stanley, Enumerative Combinatorics, Volume 2. *Cambridge Studies in Advanced Mathematics* **62**. Cambridge University Press, Cambridge, 1999.
[16] R. P. Stanley, Invariants of finite groups and their applications to combinatorics. *Bull. Amer. Math. Soc. (N.S.)* **1** (1979), 475–511.
[17] R. P. Stanley, Parking functions and noncrossing partitions, *Electron. J. Combin.* **4 (2)** (1997), R20.
[18] J. Stembridge, Some hidden relations involving the ten symmetry classes of plane partitions. *J. Combin. Theory Ser. A* **68** (1994), 372–409.
[19] B. W. Westbury, Invariant tensors and the cyclic sieving phenomenon. `arXiv:0912.1512` (2009).

E-mail address: `berget@math.ucdavis.edu`

Mathematical Sciences Building, One Shields Ave., University of California, Davis, CA 95616, USA

E-mail address: `speu@nuk.edu.tw`

No. 700 National University of Kaoshiung Rd., Nanyang District, Kaoshiung 811, Taiwan

E-mail address: `reiner@math.umn.edu`

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA